8.3 THE THEORY OF DUALITY

Chapter 2 began by saying that although the elimination technique gives one approach to $Ax = b$, a different and deeper understanding is also possible. It is exactly the same for linear programming. The mechanics of the simplex method will solve a linear program, but it is really duality that belongs at the center of the underlying theory. It is an elegant idea, and at the same time fundamental for the applications; we shall explain as much as we understand.

It starts with the standard problem:

**PRIMAL** Minimize $cx$, subject to $x \succeq 0$ and $Ax \succeq b$.

But now, instead of creating an equivalent problem with equations in place of inequalities, duality creates an entirely different problem. The dual problem starts from the same $A$, $b$, and $c$, and reverses everything. In the primal, $c$ was in the cost function and $b$ was in the constraint; in the dual, these vectors are switched. Furthermore the inequality sign is changed, and the new unknown $y$ is a row vector; the feasible set has $yA \leq c$ instead of $Ax \succeq b$. Finally, we maximize rather than minimize. The only thing that stays the same is the requirement of nonnegativity; the unknown $y$ has $m$ components, and it must satisfy $y \succeq 0$. In short, the dual of a minimum problem is a maximum problem:

**DUAL** Maximize $yb$, subject to $y \succeq 0$ and $yA \leq c$.

The dual of this problem is the original minimum problem.†

Obviously I have to give you some interpretation of all these reversals. They conceal a competition between the minimizer and the maximizer, and the explanation comes from the diet problem; I hope you will follow it through once more. The minimum problem has $n$ unknowns, representing $n$ foods to be eaten in the (nonnegative) amounts $x_1, \ldots, x_n$. The $m$ constraints represent $m$ required vitamins, in place of the one earlier constraint of sufficient protein. The entry $a_{ij}$ is the amount of the $i$th vitamin in the $j$th food, and the $i$th row of $Ax \succeq b$ forces the diet to include that vitamin in at least the amount $b_i$. Finally, if $c_j$ is the cost of the $j$th food, then $c_1x_1 + \cdots + c_nx_n = cx$ is the cost of the diet. That cost is to be minimized; this is the primal problem.

In the dual, the druggist is selling vitamin pills rather than food. His prices $y_i$ are adjustable as long as they are nonnegative. The key constraint, however, is that on each food he cannot charge more than the grocer. Since food $j$ contains vitamins in the amounts $a_{ij}$, the druggist's price for the equivalent in vitamins cannot exceed the grocer's price $c_j$. That is the $j$th constraint in $yA \leq c$. Working within this constraint, he can sell an amount $b_j$ of each vitamin for a total income of $y_jb_j + \cdots + y_nb_n = yb$—which he maximizes.

You must recognize that the feasible sets for the two problems are completely different. The first is a subset of $\mathbb{R}^n$, marked out by the matrix $A$ and the constraint vector $b$; the second is a subset of $\mathbb{R}^m$, determined by the transpose of $A$ and the other vector $c$. Nevertheless, when the cost functions are included, the two problems do involve the same input $A$, $b$, and $c$. The whole theory of linear programming hinges on the relation between them, and we come directly to the fundamental result:

**8D Duality Theorem** If either the primal problem or the dual has an optimal vector, then so does the other, and their values are the same: The minimum of $cx$ equals the maximum of $yb$. Otherwise, if optimal vectors do not exist, there are two possibilities: Either both feasible sets are empty, or else one is empty and the other problem is unbounded (the maximum is $+\infty$ or the minimum is $-\infty$).

If both problems have feasible vectors then they have optimal vectors $x^*$ and $y^*$—and furthermore $cx^* = y^*b$.

Mathematically, this settles the competition between the grocer and the druggist: The result is always a tie. We will find a similar "minimax theorem" and a similar equilibrium in game theory. These theorems do not mean that the customer pays nothing for an optimal diet, or that the matrix game is completely fair to both players. They do mean that the customer has no economic reason to prefer vitamins over food, even though the druggist guaranteed to match the grocer on every food—and on expensive foods, like peanut butter, he sells for less. We will show that expensive foods are kept out of the optimal diet, so the outcome can still be (and is) a tie.

This may seem like a total stalemate, but I hope you will not be fooled. The optimal vectors contain the crucial information. In the primal problem, $x^*$ tells the purchaser what to do. In the dual, $y^*$ fixes the natural prices (or "shadow prices") at which the economy should run. Insofar as our linear model reflects the true economy, these vectors represent the decisions to be made. They still need to be computed by the simplex method; the duality theorem tells us their most important property.

We want to start on the proof. It may seem obvious that the druggist can raise his prices to meet the grocer's, but it is not. Or rather, only the first part is: Since each food can be replaced by its vitamin equivalent, with no increase in cost, all adequate diets must be at least as expensive as any price the druggist would charge. This is only a one-sided inequality, druggist's price $\leq$ grocer's price, but it is fundamental. It is called weak duality, and it is easy to prove for any linear

† There is complete symmetry between primal and dual. We started with a minimization, but the simplex method applies equally well to a maximization—and anyway both problems get solved at once.
program and its dual:

8E. If \( x \) and \( y \) are any feasible vectors in the minimum and maximum problems, then \( yb \leq cx \).

**Proof** Since the vectors are feasible, they satisfy

\[ Ax \geq b \text{ and } yA \leq c. \]

Furthermore, because feasibility also included \( x \geq 0 \) and \( y \geq 0 \), we can take inner products without spoiling the inequalities:

\[ yAx \geq yb \text{ and } yAx \leq cx. \]

Since the left sides are identical, we have weak duality \( yb \leq cx \).

The one-sided inequality is easy to use. First of all, it prohibits the possibility that both problems are unbounded. If \( yb \) is arbitrarily large, there cannot be a feasible \( x \) or we would contradict \( yb \leq cx \). Similarly, if the minimization is unbounded—if \( cx \) can go down to \( -\infty \)—then the dual cannot admit a feasible \( y \).

Second, and equally important, we can tell immediately that any vectors which achieve equality, \( yb = cx \), must be optimal. At that point the grocer's price equals the druggist's price, and we recognize an optimal diet and optimal vitamin prices by the fact that the consumer has nothing to choose:

8F. If the vectors \( x \) and \( y \) are feasible and \( cx = yb \), then \( x \) and \( y \) are optimal.

**Proof** According to 8E, no feasible \( y \) can make \( yb \) larger than \( cx \). Since our particular \( y \) achieves this value, it is optimal. Similarly no feasible \( x \) can bring \( cx \) below the number \( yb \), and any \( x \) that achieves this minimum must be optimal.

We give an example with two foods and two vitamins. Note how \( A^T \) appears when we write out the dual, since \( yA \leq c \) for row vectors means \( A^Ty \leq c^T \) for columns.

**PRIMAL** Minimize \( x_1 + 4x_2 \)

subject to \( x_1 \geq 0, x_2 \geq 0, \)

\[ 2x_1 + x_2 \geq 6 \]

\[ 5x_1 + 3x_2 \geq 7. \]

**DUAL** Maximize \( 6y_1 + 7y_2 \)

subject to \( y_1 \geq 0, y_2 \geq 0, \)

\[ 2y_1 + 5y_2 \leq 1 \]

\[ y_1 + 3y_2 \leq 4. \]

The choice \( x_1 = 3 \) and \( x_2 = 0 \) is feasible, with cost \( x_1 + 4x_2 = 3 \). In the dual problem \( y_1 = \frac{1}{2} \) and \( y_2 = 0 \) give the same value \( 6y_1 + 7y_2 = 3 \). These vectors must be optimal.

This seems almost too simple. Nevertheless it is worth a closer look, to find out what actually happens at the moment when \( yb \leq cx \) becomes an equality. It is like calculus, where everybody knows the condition for a maximum or a minimum: The first derivatives are zero. On the other hand, everybody forgets that this condition is completely changed by the presence of constraints. The best example is a straight line sloping upward; its derivative is never zero, calculus is almost helpless, and the maximum is certain to occur at the end of the interval. That is exactly the situation that we face in linear programming! There are more variables, and an interval is replaced by a feasible set in several dimensions, but still the maximum is always found at a corner of the feasible set. In the language of the simplex method, there is an optimal \( x \) which is basic: It has only \( m \) nonzero components.

The real problem in linear programming is to decide which corner it is. For this, we have to admit that calculus is not completely helpless. Far from it, because the device of "Lagrange multipliers" will bring back zero derivatives at the maximum and minimum, and in fact the dual variables \( y \) are exactly the Lagrange multipliers for the problem of minimizing \( cx \). This is also the key to nonlinear programming. The conditions for a constrained minimum and maximum will be stated mathematically in equation (2), but first I want to express them in economic terms: The diet \( x \) and the vitamin prices \( y \) are optimal when

(i) The grocer sells zero of any food that is priced above its vitamin equivalent.

(ii) The druggist charges zero for any vitamin that is oversupplied in the diet.

In the example, \( x_2 = 0 \) because the second food is too expensive. Its price exceeds the druggist's price, since \( y_1 + 3y_2 \leq 4 \) is a strict inequality \( \frac{1}{2} + 0 < 4 \). Similarly, \( y_1 = 0 \) if the 1th vitamin is oversupplied; it is a "free good," which means it is worthless. The example required seven units of the second vitamin, but the diet actually supplied \( 5x_1 + 3x_2 = 15 \), so we found \( y_2 = 0 \). You can see how the duality has become complete; it is only when both of these conditions are satisfied that we have an optimal pair.

These **optimality conditions** are easy to understand in matrix terms. We are comparing the vector \( Ax \) to the vector \( b \) (remember that feasibility requires \( Ax \geq b \)) and we look for any components in which equality fails. This corresponds to a vitamin that is oversupplied, so its price is \( y_i = 0 \). At the same time we compare \( yA \) with \( c \), and expect all strict inequalities (expensive foods) to correspond to \( x_i = 0 \) (omission from the diet). These are the "complementary slackness conditions" of linear programming, and the "Kuhn-Tucker conditions" of nonlinear programming:

8G. **Equilibrium Theorem** Suppose the feasible vectors \( x \) and \( y \) satisfy the following complementary slackness conditions:

\[ (Ax)_i > b_i \text{ then } y_i = 0, \quad \text{and} \quad (yA)_j < c_j \text{ then } x_j = 0. \]

Then \( x \) and \( y \) are optimal. Conversely, optimal vectors must satisfy (2).
Proof The key equations are

\[ yb = y(Ax) = (yA)x = cx. \]  

(3)

Normally only the middle equation is certain. In the first equation, we are sure that \( y \geq 0 \) and \( Ax \geq b \), so we are sure of \( yb \leq y(Ax) \). Furthermore, there is only one way in which equality can hold: *Any time there is a discrepancy \( b_i < (Ax)_i \), the factor \( y_i \) that multiplies these components must be zero.* Then this discrepancy makes no contribution to the inner products, and equality is saved.

The same is true for the remaining equation: Feasibility gives \( x \geq 0 \) and \( yA \leq c \) and therefore \( yAx \leq cx \). We get equality only when the second slackness condition is fulfilled: If there is an overpricing \( (Ay)_j < c_j \), it must be canceled through multiplication by \( x_j = 0 \). This leaves us with \( yb = cx \) in the key equation (3), and it is this equality that guarantees (and is guaranteed by) the optimality of \( x \) and \( y \).

The Proof of Duality

So much for the one-sided inequality \( yb \leq cx \). It was easy to prove, it gave a quick test for optimal vectors (they turn it into an equality), and now it has given a set of necessary and sufficient slackness conditions. The only thing it has not done is to show that the equality \( yb = cx \) is really possible. Until the optimal vectors are actually produced, which cannot be done by a few simple manipulations, the duality theorem is not complete.

To produce them, we return to the simplex method—which has already computed the optimal \( x \). Our problem is to identify at the same time the optimal \( y \), showing that the method stopped in the right place for the dual problem (even though it was constructed to solve the primal). First we recall how it started. The \( m \) inequalities \( Ax \geq b \) were changed to equations, by introducing the slack variables \( w = Ax - b \) and rewriting feasibility as

\[
\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = b, \quad \begin{bmatrix} x \\ w \end{bmatrix} \geq 0.
\]

(4)

Then every step of the method picked out \( m \) columns of the long matrix \( \begin{bmatrix} A & -I \end{bmatrix} \) to be basic columns, and shifted them (at least theoretically, if not physically) to the front. This produced \( \begin{bmatrix} B & N \end{bmatrix} \), and the corresponding shift in the long cost vector \( [c \ 0] \) reordered its components into \( [c_B \ c_N] \). The stopping condition, which brought the simplex method to an end, was \( r = c_N - c_B B^{-1}N \geq 0 \).

We know that *this condition \( r \geq 0 \) was finally met*, since the number of corners is finite. At that moment the cost was

\[ cx = [c_B \ 0] \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = c_B B^{-1}b, \quad \text{the minimum cost.} \]

(5)

If we can choose \( y = c_B B^{-1} \) in the dual, then we certainly have \( yb = cx \). The minimum and maximum will be equal. Therefore, we must show that this \( y \) satisfies the dual constraints \( yA \leq c \) and \( y \geq 0 \). We have to show that

\[ y[A \ -I] \leq [c \ 0]. \]

(6)

When the simplex method resuffles the long matrix and vector to put the basic variables first, this rearranges the constraints in (6) into

\[ y[B \ N] \leq [c_B \ c_N]. \]

(7)

For our choice \( y = c_B B^{-1} \), the first half is an equality and the second half is \( c_B B^{-1}N \leq c_N \). This is the stopping condition \( r \geq 0 \) that we know to be satisfied! Therefore our \( y \) is feasible, and the *duality theorem is proved*. By locating the critical \( m \) by \( m \) matrix \( B \), which is nonsingular as long as degeneracy is forbidden, the simplex method has produced the optimal \( y^* \) as well as \( x^* \).

Shadow Prices

How does the minimum cost change if we change the right side \( b \) or the cost vector \( c \)? This is a question in *sensitivity analysis*, and it allows us to squeeze out of the simplex method a lot of the extra information it contains. For an economist or an executive, these questions about *marginal cost* are the most important. If we allow large changes in \( b \) or \( c \), the solution behaves in a very jumpy way. As the price of eggs increases, there will be a point where they disappear from the diet; in the language of linear programming, the variable \( x_{eggs} \) will jump from basic to free. To follow it properly, we would have to introduce what is called parametric programming. But if the changes are small, which is much more likely, then *the corner which was optimal remains optimal*, the choice of basic variables does not change. In other words, \( B \) and \( N \) stay the same. Geometrically, we have shifted the feasible set a little (by changing \( b \)), and we have tilted the family of planes that come up to meet it (by changing \( c \)), but if these changes are small, contact occurs first at the same (slightly moved) corner.

At the end of the simplex method, when the right choice of basic variables is known, the corresponding \( m \) columns of \( A \) make up the basis matrix \( B \). At that corner,

\[ \text{minimum cost} = c_B B^{-1}b = y^*b. \]

A shift of size \( \Delta b \) changes the minimum cost by \( y^* \Delta b \). The dual solution \( y^* \) gives the rate of change of minimum cost (its derivative) with respect to changes in \( b \). The components of \( y^* \) are the *shadow prices*, and they make sense; if the requirement for vitamin \( B_1 \) goes up by \( \Delta \), and the druggist’s price is \( y^*_1 \), and he is completely...
competitive with the grocer, then the diet cost (from druggist or grocer) will go up by \(\gamma^T \Delta\). In case \(y^T \Delta\) is zero, vitamin \(B_1\) is a "free good" and a small change in its requirement has no effect—the diet already contained more than enough.

We now ask a different question. Suppose we want to insist that the diet contain at least some small minimum amount of egg. The nonnegativity condition \(x_{\text{egg}} \geq 0\) is changed to \(x_{\text{egg}} \geq \delta\). How does this change the cost?

If eggs were in the optimal diet, there is no change—the new requirement is already satisfied and costs nothing extra. But if they were outside the diet, it will cost something to include the amount \(\delta\). The increase will not be the full price \(c_{\text{egg}}\delta\), since we can cut down on other foods and partially compensate. The increase is actually governed by the "reduced cost" of eggs—their own price, minus the price we are paying for the equivalent in cheaper foods. To compute it we return to equation (2) of Section 8.2:

\[
\text{cost} = (c_n - c_B B^{-1}N)x_N + c_B B^{-1}b = r x_N + c_B B^{-1}b.
\]

If egg is the first free variable, then increasing the first component of \(x_N\) to \(\delta\) will increase the cost by \(r_1 \delta\). Therefore, the real cost is \(r_1\). Similarly, \(r\) gives the actual costs for all other nonbasic foods—the changes in total cost as the zero lower bounds on \(x\) (the nonnegativity constraints) are moved upwards. We know that \(r \geq 0\), because this was the stopping test; and economics tells us the same thing, that the reduced cost of eggs cannot be negative or they would have entered the diet.

The Theory of Inequalities

There is more than one way to study duality. The approach we followed—to prove \(yb \leq cx\), and then use the simplex method to get equality—was convenient because that method had already been established, but overall it was a long proof. Of course it was also a constructive proof; \(x^*\) and \(y^*\) were actually computed. Now we look briefly at a different approach, which leaves behind the mechanics of the simplex algorithm and looks more directly at the geometry. I think the key ideas will be just as clear (in fact, probably clearer) if we omit some of the details.

The best illustration of this approach came in the fundamental theorem of linear algebra. The problem in Chapter 2 was to solve \(Ax = b\), in other words to find \(b\) in the column space of \(A\). After elimination, and after the four subspaces, this solvability question was answered in a completely different way by Exercise 3.1.11:

\[84\] Either \(Ax = b\) has a solution, or else there is a \(y\) such that \(yA = 0\), \(yb \neq 0\).

This is the theorem of the alternative, because to find both \(x\) and \(y\) is impossible: if \(Ax = b\) then \(yAx = yb \neq 0\), and this contradicts \(yAx = 0x = 0\). In the language of subspaces, either \(b\) is in the column space of \(A\) or else it has a nonzero component sticking into the perpendicular subspace, which is the left nullspace of \(A\). That component is the required \(y\).

For inequalities, we want to find a theorem of exactly the same kind. The right place to start is with the same system \(Ax = b\), but with the added constraint \(x \geq 0\). When does there exist not just a solution to \(Ax = b\), but a nonnegative solution? In other words, when is the feasible set nonempty in the problem with equality constraints?

To answer that question we look again at the combinations of the columns of \(A\). In Chapter 2, when any \(x\) was allowed, \(b\) was anywhere in the column space. Now we allow only nonnegative combinations, and the \(b\)'s no longer fill out a subspace. Instead, they are represented by the cone-shaped region in Fig. 8.6. For an \(m\) by \(n\) matrix, there would be \(n\) columns in \(m\)-dimensional space, and the cone becomes an open-ended pyramid. In the figure, there are four columns in two-dimensional space, and \(A\) is 2 by 4. If \(b\) lies in this cone, there is a nonnegative solution to \(Ax = b\); otherwise there is not.

Our problem is to discover the alternative: What happens if \(b\) lies outside the cone? That possibility is illustrated in Fig. 8.7, and you can interpret the geometry at a glance. There is a "separating hyperplane," which goes through the origin and has the vector \(b\) on one side and the whole cone on the other side. (The prefix hyper is only to emphasize that the number of dimensions may be large; the plane consists, as always, of all vectors perpendicular to a fixed vector \(y\).) The inner product between \(y\) and \(b\) is negative since they make an angle greater than 90°, whereas the inner product between \(y\) and every column of \(A\) is positive. In matrix terms this means that \(yb < 0\) and \(yA \geq 0\), which is the alternative we are looking for.

\[85\] You see that this proof is not constructive! We only know that a component of \(b\) must be in the left nullspace, or \(b\) would have been in the column space.
would be positive, whereas orthogonal vectors have inner product zero. On the other hand, it is not quite certain that either \( S \) or \( S^\perp \) has to contain a positive vector. \( S \) might be the \( x \) axis and \( S^\perp \) the \( y \) axis, in which case they contain only the "semipositive" vectors [1 0] and [0 1]. What is remarkable is that this slightly weaker alternative does work. Either \( S \) contains a positive vector \( x \), or \( S^\perp \) contains a semipositive vector \( y \). When \( S \) and \( S^\perp \) are perpendicular lines in the plane, it is easy to see that one or the other must enter the first quadrant; but I do not see it very clearly in higher dimensions.

For linear programming, the important alternatives come when the constraints are inequalities rather than equations:

**8J** Either \( Ax \geq b \) has a solution with \( x \geq 0 \), or else there is a vector \( y \) such that \( yA \geq 0 \), \( yb < 0 \), \( y \leq 0 \).

8J follows easily from 8I, using the slack variables \( w = Ax - b \) to change the inequality into an equation:

\[
\begin{bmatrix}
A & -I
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}
= b.
\]

If this has no solution with \( x \geq 0 \) and \( w \geq 0 \), then by 8I there must be a \( y \) such that \( y[A - I] \geq [0 0] \), \( yb < 0 \).

This is exactly the other alternative in 8J. It is this result that leads to a "nonconstructive proof" of the duality theorem. But we promised to stick to the geometry and omit the algebraic details, so we keep that promise.

**EXERCISES**

8.3.1 What is the dual of the following problem: Minimize \( x_1 + x_2 \), subject to \( x_1 \geq 0 \), \( x_2 \geq 0 \), \( 2x_1 + 4 \), \( x_1 + 3x_2 \geq 11 \)? Find the solution to both this problem and its dual, and verify that minimum equals maximum.

8.3.2 What is the dual of the following problem: Maximize \( y_2 \), subject to \( y_1 \geq 0 \), \( y_2 \geq 0 \), \( y_1 + y_2 \leq 3 \)? Solve both this problem and its dual.

8.3.3 Suppose \( A \) is the identity matrix (so that \( m = n \)) and the vectors \( b \) and \( c \) are nonnegative. Explain why \( x^* = b \) is optimal in the minimum problem, find \( y^* \) in the maximum problem, and verify that the two values are the same. If the first component of \( b \) is negative, what are \( x^* \) and \( y^* \)?

8.3.4 Construct a 1 by 1 example in which \( Ax \geq b \), \( x \geq 0 \) is unfeasible, and the dual problem is unbounded.
8.3.5 Starting with the 2 by 2 matrix \( A = \begin{bmatrix} 0 & 9 \\ 1 & 1 \end{bmatrix} \), choose \( b \) and \( c \) so that both of the feasible sets \( Ax \geq b, x \geq 0 \) and \( yA \leq c, y \geq 0 \) are empty.

8.3.6 If all entries of \( A \), \( b \), and \( c \) are positive, show that both the primal and the dual are feasible.

8.3.7 Show that \( x = (1, 1, 1, 0) \) and \( y = (1, 1, 0, 1) \) are feasible in the primal and dual, with

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}
\]

Then, after computing \( cx \) and \( yb \), explain how you know they are optimal.

8.3.8 Verify that the vectors in the previous exercise satisfy the complementary slackness conditions (2), and find the one slack inequality in both the primal and the dual.

8.3.9 Suppose that \( A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b = \begin{bmatrix} -1 \end{bmatrix} \), and \( c = \begin{bmatrix} 1 \end{bmatrix} \). Find the optimal \( x \) and \( y \), and verify the complementary slackness conditions (as well as \( yb = cx \)).

8.3.10 If the primal problem is constrained by equations instead of inequalities—Minimize \( cx \) subject to \( Ax = b \) and \( x \geq 0 \)—then the requirement \( y \geq 0 \) is left out of the dual: Maximize \( yb \) subject to \( yA \leq c \). Show that the one-sided inequality \( yb \leq cx \) still holds. Why was \( y \geq 0 \) needed in (1) but not here? This weak duality can be completed to full duality.

8.3.11 (a) Without the simplex method, minimize the cost \( 5x_1 + 3x_2 + 4x_3 \) subject to \( x_1 + x_2 + x_3 \geq 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \).

(b) What is the shape of the feasible set?

(c) What is the dual problem, and what is its solution \( y \)?

8.3.12 If the primal has a unique optimal solution \( x^* \), and then \( c \) is changed a little, explain why \( x^* \) still remains the optimal solution.

8.3.13 If steak costs \( c_1 = \$3 \) and peanut butter \( c_2 = \$2 \), and they give two units and one unit of protein (four units are required), find the shadow price of protein and the reduced cost of peanut butter.

8.3.14 If \( A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), describe the cone of nonnegative combinations of the columns. If \( b \) lies inside that cone, say \( b = (3, 2) \), what is the feasible vector \( x \)? If \( b \) lies outside, say \( b = (0, 1) \), what vector \( y \) will satisfy the alternative?

8.3.15 In three dimensions, can you find a set of six vectors whose cone of nonnegative combinations fills the whole space? What about four vectors?

8.3.16 Use 8H to show that there is no solution (because the alternative holds) to

\[
\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

8.3.17 Use 8I to show that there is no nonnegative solution (because the alternative holds) to

\[
\begin{bmatrix} 1 & 3 & -5 \\ 1 & -4 & -7 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.
\]

8.3.18 Show that the alternatives in 8J (\( Ax \geq b, x \geq 0, yA \geq c, y \leq 0 \)) cannot both hold. Hint: \( yA \).