Cohen–Grossberg Theorem
CN550 Lecture 11
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Cohen and Grossberg (1983)

Absolute stability of global pattern formation and parallel memory storage by competitive neural networks

Cohen Grossberg Theorem

Global pattern formation: $\exists$ limits

$$\lim_{t \to \infty} (x_1(t), \ldots, x_n(t))$$

for all $(x_1(0), \ldots, x_n(0))$

Absolute stability: global pattern formation for ANY choice of parameters
Method: Liapunov functions

Liapunov functions are used to prove the stability of a certain fixed point in a dynamical system.

System (autonomous)

\[ \dot{x}_i = F_i(x_1 \ldots x_n) \quad (*) \]

\( V(x_1, \ldots, x_n) \) [differentiable; \( \geq A \): bounded below] is a Liapunov function for (*) if \( V \) decreases along solutions:

\( V(x_1(t), \ldots, x_n(t)) \) is a decreasing function if \((x_1(t), \ldots, x_n(t))\) is a solution of (*)

Chain Rule:

\[ \frac{dV}{dt} = \sum_i \frac{dV}{dx_i} \cdot \frac{dx_i}{dt} = \sum_i \frac{dV}{dx_i} F_i = \nabla V \cdot \vec{F} \]

\( \nabla V \cdot \vec{F} \)

gradient
LaSalle Invariance Principle

If \( V(\vec{x}) \) is a Liapunov function for (an admissible set of) trajectories, then, as \( t \to \infty \) solutions either

\( \to \) the largest invariant set contained in:

\[
\{ \vec{x} : \frac{dV}{dt}(\vec{x}) = 0 \}
\]

\( \to \infty \) (in \( \geq 1 \) variable)
Dissipation of energy

Liapunov function generalizes the notion of dissipation of energy

**Without friction**

\[ V(\vec{x}) = \text{kinetic energy} + \text{potential energy} = \text{constant} \]

\[ \dot{V} = 0 \]

Conservative system (energy conserved)

**With friction**

Energy = \[ V(\vec{x}) \]

\[ V(\vec{x}) \geq 0 \]

\[ \dot{V} < 0 \]

when \( \vec{x} \neq \vec{0} \)

Dissipation of total energy in system \( \rightarrow \) limit
Problems with Liapunov functions

\[ \dot{x}_i = F_i(x_1...x_n) \quad (*) \]

1. There exists too many

\[ V(\bar{x}) = 0 \]

\[ \frac{dV}{dt} = \sum_i \frac{dV}{dx_i} \frac{dx_i}{dt} \]

\[ = \sum_i 0 \cdot F_i(\bar{x}) = 0 \]

All solutions \( \rightarrow \) to some (set of) point(s) \( \bar{x} \) or \( \rightarrow \infty \) [trivial]

2. There exists too few

Usually difficult (impossible)

to construct useful \( V(\bar{x}) \)

from \( F_1...F_n \)

When possible, a powerful method
Converse of 2

Relatively easy…

Given $V(\bar{x}) \geq 0$ construct a system (*) for which $V(\bar{x})$ is a Liapunov function

Need: $\dot{V} = \sum_i \frac{dV}{dx_i} \frac{dx_i}{dt} \leq 0$

Let: $\frac{dx_i}{dt} = \dot{x}_i = -\alpha_i(x) \frac{dV}{dx_i}$ [see backprop cost fn]

Solutions $\rightarrow$ [the largest invariant set contained in]

$$\{\bar{x} : \alpha_1 \frac{dV}{dx_1} = \ldots = \alpha_n \frac{dV}{dx_n} = 0\}$$

or $\rightarrow \infty$

$V(\bar{x})$ is a cost function
Hypotheses of Cohen Grossberg theorem are needed to...

1. Construct a Liapunov function
   \[ V(\vec{x}) \]

2. Ensure that admissible trajectories remain bounded
   \[ x_i(0) > 0 \]

3. Ensure that
   \[ E = \{ \vec{x} : \dot{V}(\vec{x}) = 0 \} \]

4. Rule out problem cases
   e.g. consists of all critical points
Proof of hypothesis for $\dot{V}(\vec{x}) \leq 0$

system: \[ \dot{x}_i = a_i(x_i)[b_i(x_i) - \sum_{k} c_{ik}d_k(x_k)] \]

\[ V(\vec{x}) = -\sum_{i=1}^{n} \int_{0}^{x_i} b_i(\xi_i)d_i'(\xi_i)d\xi_i + \frac{1}{2} \sum_{j,k=1}^{n} c_{jk}d_j(x_j)d_k(x_k) \]

(analogous to physical dissipation systems)

\[
\frac{dV}{dt} = \sum_{i=1}^{n} \frac{dV}{dx_i} \cdot a_i(x_i)[b_i(x_i) - \sum_{k=1}^{n} c_{ik}d_k(x_k)]
\]

where

\[
\frac{dV}{dx_i} = -b_i(x_i)d_i'(x_i) + \frac{1}{2} \sum_{k=1}^{n} c_{ik}d_i'(x_i)d_k(x_k) + \frac{1}{2} \sum_{j=1}^{n} c_{ji}d_j(x_j)d_i'(x_i)
\]

\[ j=i \]

\[ k=i \]
Proof of hypothesis for $\dot{V}(\vec{x}) \leq 0$

$$\dot{x}_i = a_i(x_i)[b_i(x_i) - \sum_k c_{ik}d_k(x_k)]$$

continued...

$$\frac{dV}{dx_i} = -d_i'(x_i)[b_i(x_i) - \frac{1}{2} \sum_{k=1}^n (c_{ik} + c_{ki})d_k(x_k)]$$

$$\frac{dV}{dt} \leq 0 \quad \text{if} \quad a_i(x_i) \geq 0$$

$$d_i'(x_i) \geq 0$$

$$c_{ik} = c_{ki}$$

$$\dot{V}(\vec{x}) = -\sum_{i=1}^n d_i'(x_i)a_i(x_i)[b_i(x_i) - \sum_{k=1}^n c_{ik}d_k(x_k)]$$
Hypotheses for $\dot{V}(\bar{x}) \leq 0$

1. Symmetry
   
   $$c_{ik} = c_{ki}$$

2. Nonnegativity
   
   $$a_i(w) \geq 0 \quad \text{for} \quad w \geq 0$$

3. Monotonicity
   
   $$d_i(w) \text{ is differentiable and } d_i'(w) \geq 0 \quad \text{for} \quad w \geq 0$$
More hypotheses for $\dot{V}(\bar{x})$ trivial

4. $c_{ik} \geq 0$

5. $a_i(w)$ continuous for $w \geq 0$ and $b_i(w)$ continuous for $w > 0$

6. $a_i(w) > 0$ for $w > 0$ and $d_i(w) \geq 0$ for $-\infty < w < \infty$