

CN510: Principles and Methods of Cognitive and Neural Modeling

Modeling of Synaptic Dynamics

Lecture 11

Instructor: Anatoli Gorchetchnikov <anatoli@bu.edu>

Conductance-based vs Current-based

The basic formalism from equivalent circuit is

$$C_m \frac{dV}{dt} = \sum I$$

Here we can consider more detailed model based on conductances

$$C_M \frac{dV}{dt} = g_i (V_m - E_i) + \dots$$

or a more simplified one where we compute the currents or current densities directly

$$C_m \frac{dV}{dt} = \sum I_i \qquad C_M \frac{dV}{dt} = \sum J_i$$

Note that the second model is additive rather than shunting

Conductance-based vs Current-based

Current-based models

- Easier to compute
- Often looks sufficient especially for leaky IaF and other simplified spike generators

Conductance-based models

- More realistic
- Might carry over some properties of shunting networks that we have seen

Nothing prevents you from modeling some currents in your neuron directly while other currents through a conductance-based mechanism (or lumping together multiple currents into a spike-generating mechanism) as long as you **watch the units**

Some Considerations

Simplest Leaky IaF has 0 as resting potential, but you can change it either through additive scaling of the V_m itself or shunting scheme on the derivative

$$C_m \frac{dV_m}{dt} = -g_L (V_m - V_{rest}) + \sum I_i$$

Quadratic IaF has to scale quadratic term so that 1 of the quadratic term is close to spiking threshold voltage

$$C_m \frac{dV_m}{dt} = AV_m^2 - C + \sum I_i$$

It also can be scaled to have resting potential at 0

$$C_m \frac{dV_m}{dt} = AV_m^2 - BV + \sum I_i$$

More Considerations

If you need detailed Na or K voltage gated channels, then probably it is better to do Hodgkin-Huxley spiking rather than any of IaF mechanisms

If you decide to use Izhikevich simple model for spiking, maybe it is better to use his parameter manipulations to lump in other voltage-dependent currents

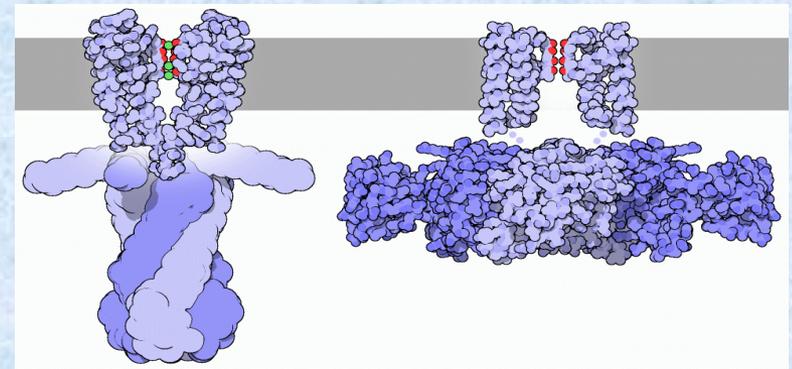
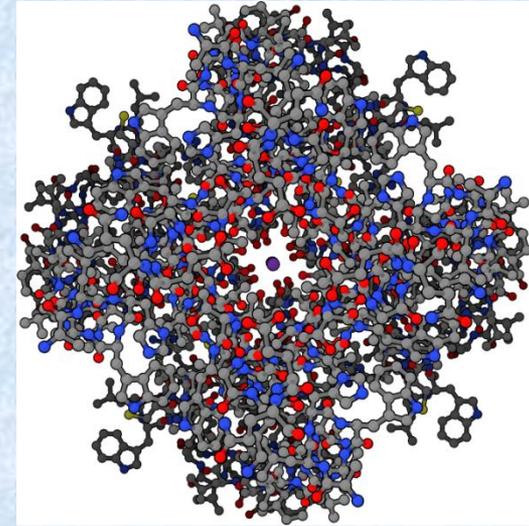
One of the approaches would be to model a single cell in detail, look at responses to a number of simple inputs and pick Izhikevich parameter set that provides the same response

Conductance of the Channels

Individual channel can be either open or closed

Conductance is usually written as a product of:

- a maximal conductance when all channels are open \bar{g}_i and
- a probability that the channel is open or the fraction of open channels $0 < g_i(V_m, t, \dots) < 1$ sometimes called gating variable



Conductance of the Channels

For voltage gated channels we use some variation of HH formalism:

$$\frac{dg}{dt} = \alpha(1-g) - \beta g$$

where $\alpha(V_m)$ is an opening rate function and $\beta(V_m)$ is a closing rate function

For chemically gated channels we can use the same formalism

Closing rate β represents unbinding of the transmitter from the receptor and usually is modeled as constant

Opening rate α is proportional to the concentration of the transmitter in the cleft

Simple Model of a Synapse

A simplest model of transmitter concentration

$$\frac{dg}{dt} = \alpha(1 - g) - \beta g$$

It is just the leaky integrator

We can simplify by splitting raise ($\alpha=1$) and fall ($\alpha=0$)

$$\frac{dg}{dt} = \alpha - (\alpha + \beta)g \quad \text{is} \quad \frac{dg}{dt} = 1 - (1 + \beta)g \quad \text{or} \quad \frac{dg}{dt} = -\beta g$$

To further simplify the modeling we can assume $\beta=0$ for the opening part

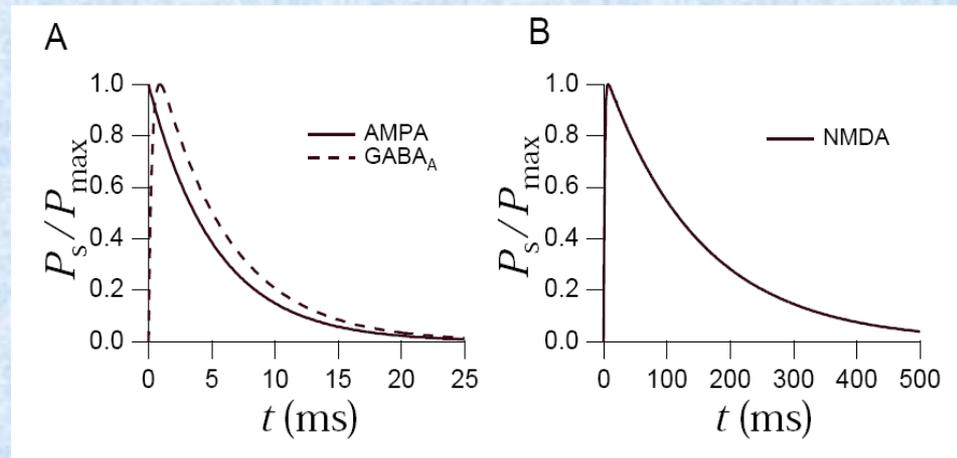
$$\frac{dg}{dt} = 1 - g \qquad \frac{dg}{dt} = -\beta g$$

Note that we can use the same formalism to model current rather than conductance (but remember the scale and timing)

Exponential Synapse

For some synapses the rise time constant is short and can be neglected

Then the whole synapse can be modeled as an instant increase in conductance (or current) followed by an exponential decay



Important feature of this synapse: it does not require temporal component of the concentration in the cleft, instantaneous spike event from presynaptic cell is sufficient

What If We Want Slower Rise?

Using equations

$$\frac{dg}{dt} = 1 - g \qquad \frac{dg}{dt} = -\beta g$$

we can model any synapse, but ideally we do not want to switch dynamics on the fly for something as abundant as synapse

Keeping the original equation

$$\frac{dg}{dt} = \alpha(1 - g) - \beta g$$

is feasible, but we might want the simplicity of exponential synapse with slower rise dynamics:

Use event triggered difference of exponentials

Equation

$$\frac{dg}{dt} = \alpha(1 - g) - \beta g$$

when the time of concentration pulse is sufficient to reach the saturation, can be approximated with an event triggered (critically damped) pendulum equation

$$\frac{d^2 g}{dt^2} = A \frac{dg}{dt} + Bg$$

For the case of different raise and fall time constants the solution is dual-exponential or beta function, for the case of identical time constants it is alpha function

Alpha and Beta Synapses

Here we ensure that the conductance reaches 1 at the peak

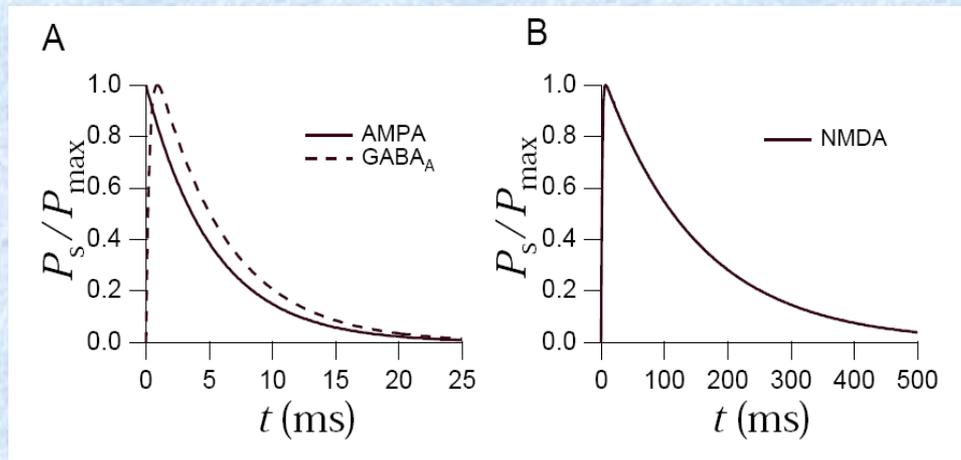
Also we do not need to worry about timing of concentration

Finally, we can reduce communication between neurons to 1 bit

$$g_i(t) = \begin{cases} \frac{p}{\tau_f - \tau_r} \left(e^{-\frac{t-t_s}{\tau_f}} - e^{-\frac{t-t_s}{\tau_r}} \right) & \text{if } \tau_f \neq \tau_r \\ \frac{t}{\tau_f} e^{-\frac{t-t_s}{\tau_f}} & \text{otherwise} \end{cases}$$

so that

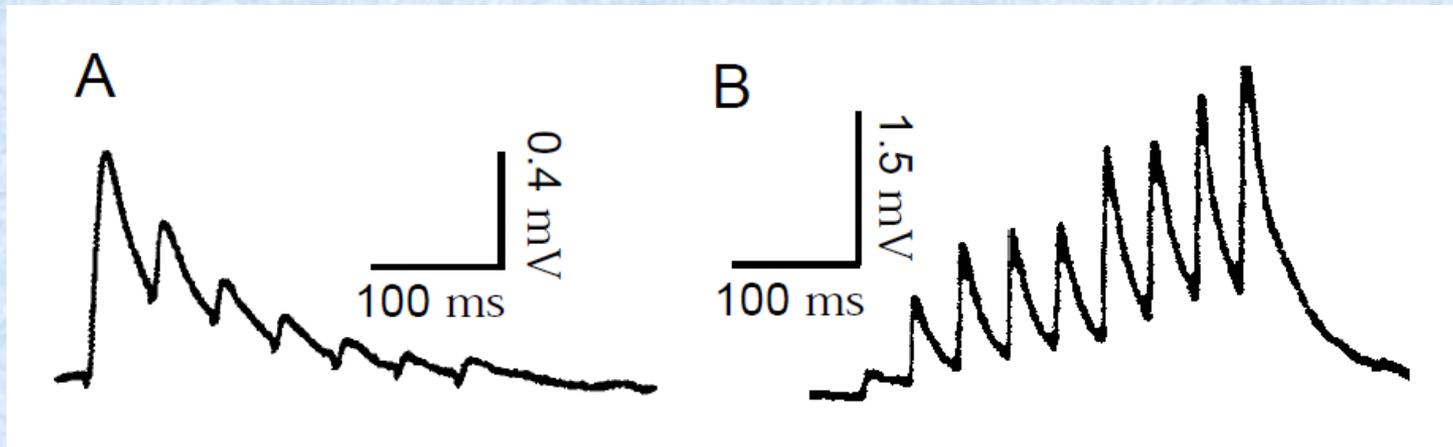
$$\max \left(\frac{p}{\tau_f - \tau_r} \left(e^{-\frac{t-t_s}{\tau_f}} - e^{-\frac{t-t_s}{\tau_r}} \right) \right) = 1$$



Presynaptic Component

General form of the conductance that we use is $\bar{g}_{ch} g_1^n g_2^k \dots$

Thus we can add (or rather multiply) presynaptic component to our conductance as well as synaptic efficacy to this formalism (note that efficacy is not bounded between 0 and 1)



Presynaptic component can represent the short term change in the probability of transmitter release (depression or facilitation)

Short-term Depression and Facilitation

Depression: release probability instantaneously drops during spike, then recovers with some time constant: similar mechanism as exponential synapse

$$\frac{dp_r}{dt} = A(1 - p_r) \quad p_r = p_r - B$$

The difference is that now the rest value is 1, and the instant change is negative and partial

Facilitation has a rest value above 0 but below 1, each spike pushes the probability up and then it relaxes

$$\frac{dp_r}{dt} = -A(p_r - p_0) \quad p_r = p_r + B$$

Note that aside from alpha and beta synapses they are all leaky integrators of some kind

Rotter-Diesmann Method

For linear differential equations and systems of equations

- Use exact analytical solution for computation
- Use initial values and compute new values at a given time
- Use new values as initial values and repeat the process for as long as necessary

Example:

For $\frac{dg}{dt} = -Ag$ solution is $g = g_o e^{-\frac{t-t_0}{\tau}}$ (with $\tau=1/A$)

Thus $g_{n+1} = g_n e^{-\frac{t_{n+1}-t_n}{\tau}}$

Rotter-Diesmann Method

For a leaky integrator with **constant** input:

$$\frac{dg}{dt} = -Ag + I$$

the solution is

$$g(t) = g_0 e^{-At} + I \int_0^t e^{-A\tau} d\tau = g_0 e^{-At} + I \left(\frac{1}{-A} e^{-At} - \frac{1}{-A} e^0 \right) = g_0 e^{-At} + \frac{I}{A} (1 - e^{-At})$$

which can be written as an iterative procedure as

$$g_{n+1} = g_n e^{-\frac{t_{n+1}-t_n}{\tau}} + I\tau \left(1 - e^{-\frac{t_{n+1}-t_n}{\tau}} \right)$$

(with $\tau = I/A$)

Rotter-Diesmann Method

For a leaky integrator with **spiking** input:

$$\frac{dg}{dt} = -Ag + x(t) \quad g = g_o e^{-\frac{t-t_0}{\tau}} + \int x(t)dt$$

If we consider an input spike train as a sequence of Dirac delta functions

$$x(t) = \sum x_n \delta(t - t_n)$$

then the integral at step $n+1$ is just x_{n+1} , so

$$g_{n+1} = g_n e^{-\frac{t_{n+1}-t_n}{\tau}} + x_{n+1}$$

Note: if before we did not care when t_{n+1} is, now we want it to coincide with presynaptic spike

RD Standalone Exponential Synapse

Equation

$$g_{n+1} = g_n e^{-\frac{t_{n+1}-t_n}{\tau}} + x_{n+1}$$

models exponential synapse. Furthermore, if our x_{n+1} is computed through a similar equation, then we get exponential synapse with short-term facilitation or depression

The only difference for facilitation/depression would be a slight change in the solution form:

$$\frac{dp_r}{dt} = -A(p_r - p_0) \quad p_{n+1} = p_n e^{-\frac{t_{n+1}-t_n}{\tau}} + p_0 \left(1 - e^{-\frac{t_{n+1}-t_n}{\tau}} \right) + x_{n+1}$$

so that it relaxes to p_0 rather than to 0

Leaky IaF with Delta Synapse

The same equation

$$V_{n+1} = V_n e^{-\frac{t_{n+1}-t_n}{\tau}} + x_{n+1}$$

can also be used for the simplest model of the leaky IaF with delta synapse

Delta synapse simply busts the membrane potential instantaneously by a certain amount when the spike arrives

Again, we can add short-term plasticity if we compute x_{n+1} like before

Furthermore, here we only need to compute voltage when the input arrives: possibility for pure event-driven system

Reminder of RD Exact Integration

Leaky integrator: $V_{n+1} = V_n e^{-\frac{t_{n+1}-t_n}{\tau}}$

Leaky integrator with constant input:

$$\dot{V} = -AV + I$$

$$V(t) = V_0 e^{-At} + I \int_0^t e^{-A\tau} d\tau = V_0 e^{-At} + I \left(\frac{1}{-A} e^{-At} - \frac{1}{-A} 0 \right) = V_0 e^{-At} + \frac{I}{A} (1 - e^{-At})$$

$$V_{n+1} = V_n e^{-A(t_{n+1}-t_n)} + \frac{I}{A} (1 - e^{-A(t_{n+1}-t_n)})$$

Leaky integrator with spiking input through delta synapse:

$$V_{n+1} = V_n e^{-\frac{t_{n+1}-t_n}{\tau}} + x_{n+1}$$

Exponential synapse: $I_{n+1} = I_n e^{-\frac{t_{n+1}-t_n}{\tau}} + x_{n+1}$ or $g_{n+1} = g_n e^{-\frac{t_{n+1}-t_n}{\tau}} + x_{n+1}$

RD Alpha and Beta Synapses

$$\frac{d^2 g}{dt^2} = A \frac{dg}{dt} + Bg$$

Take the equation

$$\ddot{g} = -(a+b)\dot{g} - abg + x(t)$$

and convert it to a system of two first-order equations

$$\begin{cases} -a\dot{g} + \dot{y} = -a(-ag + y) - b(-ag + y) - abg + x(t) \\ \dot{g} = -ag + y \end{cases}$$

$$\begin{cases} a^2 g - ay + \dot{y} = a^2 g - ay + abg - by - abg + x(t) \\ \dot{g} = -ag + y \end{cases} = \begin{cases} \dot{y} = -by + x(t) \\ \dot{g} = -ag + y \end{cases}$$

now we can provide the exact solution for this system in terms of matrix exponentials

RD Alpha and Beta Synapses

$$\begin{cases} \dot{y} = -by + x(t) \\ \dot{g} = -ag + y \end{cases}$$

This solution is

$$G_{n+1} = G_n e^{M(t_{n+1}-t_n)} + X_{n+1}$$

$$\begin{bmatrix} y \\ g \end{bmatrix}_{n+1} = \begin{bmatrix} y \\ g \end{bmatrix}_n^T e^{M(t_{n+1}-t_n)} + \begin{bmatrix} x \\ 0 \end{bmatrix}_{n+1}$$

Here the matrix M for alpha synapse is

$$e^{M(t_{n+1}-t_n)} = \begin{bmatrix} e^{-\frac{t_{n+1}-t_n}{\tau}} & 0 \\ (t_{n+1}-t_n)e^{-\frac{t_{n+1}-t_n}{\tau}} & e^{-\frac{t_{n+1}-t_n}{\tau}} \end{bmatrix} \begin{matrix} y \\ g \end{matrix}$$

RD Alpha and Beta Synapses

$$\begin{cases} \dot{y} = -by + x(t) \\ \dot{g} = -ag + y \end{cases}$$

This solution is

$$G_{n+1} = G_n e^{M(t_{n+1}-t_n)} + X_{n+1}$$

$$\begin{bmatrix} y \\ g \end{bmatrix}_{n+1} = \begin{bmatrix} y \\ g \end{bmatrix}_n^T e^{M(t_{n+1}-t_n)} + \begin{bmatrix} x \\ 0 \end{bmatrix}_{n+1}$$

The matrix M for beta synapse is

$$e^{M(t_{n+1}-t_n)} = \begin{bmatrix} \begin{matrix} \mathbf{y} \\ \mathbf{g} \end{matrix} \\ e^{-\frac{t_{n+1}-t_n}{\tau_r}} & 0 \\ \frac{\tau_r \tau_f}{\tau_f - \tau_r} \begin{pmatrix} e^{-\frac{t_{n+1}-t_n}{\tau_f}} & -e^{-\frac{t_{n+1}-t_n}{\tau_r}} \end{pmatrix} & e^{-\frac{t_{n+1}-t_n}{\tau_f}} \end{bmatrix} \begin{matrix} \mathbf{y} \\ \mathbf{g} \end{matrix}$$

Leaky Integrator with Exponential Synapse

Again, a system of two equations:
$$\begin{cases} \dot{I} = -AI + x(t) \\ \dot{V} = -BV + I \end{cases}$$

Note that it is identical to the synaptic system we just considered:
$$\begin{cases} \dot{y} = -by + x(t) \\ \dot{g} = -ag + y \end{cases}$$

So not only we can apply the exact same matrices, but we also can claim that the voltage here will follow the pendulum equation

But only if we consider **synapse in a current form**

Conductance based synapse introduces nonlinearity to the system and we cannot apply RD directly

Leaky Integrator with Alpha/Beta Synapses

Now we have a system of three equations:

$$\begin{cases} \dot{y} = -Ay + x(t) \\ \dot{I} = -BI + y \\ \dot{V} = -CV + I \end{cases}$$

Not much more difficult:
The matrix becomes 3x3

$$e^{M(t_{n+1}-t_n)} = \begin{matrix} & \mathbf{y} & & \mathbf{I} & & \mathbf{V} \\ \begin{bmatrix} e^{-\frac{t_{n+1}-t_n}{\tau_s}} & 0 & 0 \\ (t_{n+1}-t_n)e^{-\frac{t_{n+1}-t_n}{\tau_s}} & e^{-\frac{t_{n+1}-t_n}{\tau_s}} & 0 \\ \left(\frac{\tau_s \tau_m}{\tau_m - \tau_s}\right)^2 \begin{pmatrix} e^{-\frac{t_{n+1}-t_n}{\tau_m}} & -e^{-\frac{t_{n+1}-t_n}{\tau_s}} \end{pmatrix} - \\ -\frac{\tau_s \tau_m}{\tau_m - \tau_s} (t_{n+1}-t_n)e^{-\frac{t_{n+1}-t_n}{\tau_s}} & \frac{\tau_s \tau_m}{\tau_m - \tau_s} \begin{pmatrix} e^{-\frac{t_{n+1}-t_n}{\tau_m}} & -e^{-\frac{t_{n+1}-t_n}{\tau_s}} \end{pmatrix} & e^{-\frac{t_{n+1}-t_n}{\tau_m}} \end{bmatrix} \end{matrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

Converting These to LIAF

Need to monitor the threshold crossing for the postsynaptic cell

Easy for delta and exponential synapse since membrane potential only goes up at the arrival times of presynaptic spikes

Both beta and alpha synapses provide delay before the input is the strongest, so need to watch the voltage on the upswing of input curve

Or we can use the equations to precompute the timing when the voltage will cross the threshold and recompute it on every input arrival

Next Time

Learning laws described at the network level and based on neurophysiological findings.

Concepts of local and global learning rules.

A set of Hebbian rules is analyzed for stability and properties of resulting weights

Readings:

- D&A Chapter 8 (section 1-3).
- Vasilkoski, Z. et al. (2011). Review of stability properties of neural plasticity rules for implementation on memristive neuromorphic hardware